The study of the motion of a gyroscope without rotational symmetry

F-C Ciornei[1], S Alaci[1], R D Pentiuc[1], C Bujoreanu[2] and O T Rusu[1]
[1]"Stefan cel Mare" University, Suceava, Romania
[2]"Gheorghe Asachi" Technical University, Iasi, Romania

florina.ciornei@usm.ro

Abstract. In rigid dynamics theory the gyroscope has a special role, as one of the bodies with the most complex motion, having three degrees of freedom. As consequence of the complexity of the motion is the fact that only for three situations, the equations of motion can be analytically integrated. For all the three situations, it is assumed that the principal moments of inertia (PMI) with respect to the principal axes that are normal to the rotation axes, are equal, condition fulfilled by the shape of body of revolution of the gyroscope. The paper analyses the case when the two principal moments of inertia are unequal. The equations of motion are numerically integrated for different values of the ratio between the two PMI. The effect of the disparity between the two PMI upon the motion of the gyroscope is proved by comparison with the motion of the axisymmetric gyroscope, both obtained by numerical integration.

1. Introduction

In figure 1, the motion of the rigid body is referred to the immobile Cartesian coordinate system $Oxyz$ of versors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$. The frame attached to the rigid body is also a tri-orthogonal coordinate system $Ox'y'z'$ with versors $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$. The position vector of the point $\boldsymbol{M}$ is expressed by the relation:

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{r}' \tag{1}$$

where $\boldsymbol{r}_0$ is the position vector of the origin of the mobile frame. Under the form (1) the relation cannot be applied since the vectors from the right member are expressed in different coordinate systems. The explicit relation is:

$$x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = x_0\boldsymbol{i} + y_0\boldsymbol{j} + z_0\boldsymbol{k} + x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'\boldsymbol{j}' \tag{2}$$

Applying the scalar product for the equation (2) and the versors of the immobile frame, the following matrix form is obtained:

$$[x \quad y \quad z]^T = [x_O \quad y_O \quad z_O]^T + \boldsymbol{A}[x' \quad y' \quad z']^T \tag{3}$$

In a concentrated manner, the equation (3) is written as:

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{A}\boldsymbol{r}' \tag{4}$$

where $\boldsymbol{A}$ is an orthogonal matrix $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}_3$, $A_{ij}A_{jk} = \delta_{ik}$, [1] describing the position of the axes (x', y', z') with respect to (x, y, z) x', y', z'), where δ_{ik} is the Kroneker symbol.

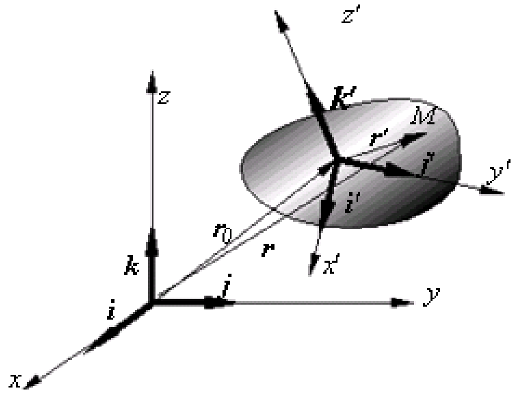

Figure 1. Position of a point of a rigid body

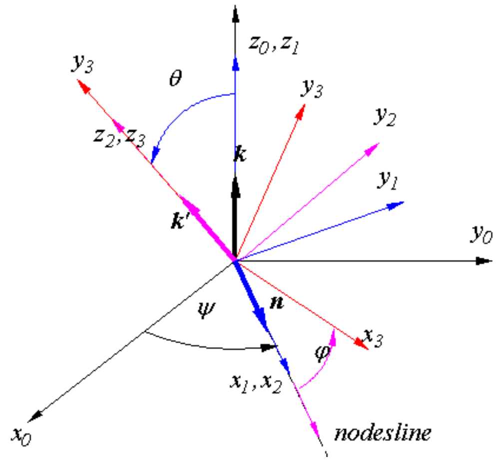

Figure 2. Over posing two coordinate systems

As a conclusion, *six parameters are required* for stipulation of the position of the rigid: three needed for specifying the position of the origin of the mobile system and the other three for specifying the orientation of the axes of the mobile system with respect to the axes of the fixed frame. Regarding the first three parameters, they are the projections of the $\boldsymbol{r}_0$ vector on the axes of the fixed system, but the identification of the last three parameters is more difficult. In order to overlap two systems having the same origin, there are required three finite rotations about three well précised axes, as shown by Euler [2]. Thus, the $\boldsymbol{A}$ matrix may be obtained as the product of three matrices, corresponding to the three rotations [3]. According to Euler [2], the line of the nodes is the intersection between the planes x_0y_0 and x_1y_1. The three rotations are: rotation (*precession*) around z_0 axis of $\boldsymbol{k}$ versor, with the ψ angle, of the system $''0''$ about the system $''1''$; rotation (*nutation*) about Ox_1 axis, of $\boldsymbol{n}$ versor, with the ψ angle and rotation (*self-rotation*) about Oz_2 axis, of $\boldsymbol{k}'$ versor. After applying the three successive rotations, the 0 system will finally get into the 3 position. By denoting with Ω the infinitesimal vector obtained as the sum of the three infinitesimal rotations:

$$d\Omega = \boldsymbol{k}d\psi + \boldsymbol{n}\,d\theta + \boldsymbol{k}'\,d\varphi \tag{5}$$

it will be on its turn, an infinitesimal rotation which applied to a vector will produce a vector placed in a plane normal to the direction of $d\,\Omega$ and the rotation angle is given by the modulus of the vector (5). By dividing the relation (5) to dt, a ω vector is obtained:

$$\omega = \dot{\psi}\,\boldsymbol{k} + \dot{\theta}\,\boldsymbol{n} + \dot{\varphi}\,\boldsymbol{k}' \tag{6}$$

With the product used in finding the $\boldsymbol{A}$ matrix, there can be obtained the projections of any versor of an axis on the axes of another system.

2. The equations of motion for the rigid body with a fixed point

The rigid body with a fixed point (spherical joint) is represented in figure 3. The unknowns of the problem are the expressed using the three Euler angles and the components of the reaction from the spherical pair. The position of the rigid are obtained from Euler equations, where the mobile axes are the principal axes of inertia, passing through the fix point:

$$\begin{bmatrix} J'_x & 0 & 0 \\ 0 & J'_y & 0 \\ 0 & 0 & J'_z \end{bmatrix}\begin{bmatrix} \dot{\omega}'_x \\ \dot{\omega}'_y \\ \dot{\omega}'_z \end{bmatrix}+\begin{bmatrix} 0 & -\omega'_z & \omega'_y \\ \omega'_z & 0 & -\omega'_x \\ -\omega'_y & \omega'_x & 0 \end{bmatrix}\begin{bmatrix} J'_x & 0 & 0 \\ 0 & J'_y & 0 \\ 0 & 0 & J'_z \end{bmatrix}\begin{bmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{bmatrix}=\begin{bmatrix} M'_{Ox} \\ M'_{Oy} \\ M'_{Oz} \end{bmatrix} \Leftrightarrow \begin{matrix} J'_x \dot{\omega}'_x + (J'_z - J'_y)\omega'_y \omega'_z = M'_{Ox} \\ J'_y \dot{\omega}'_y + (J'_x - J'_z)\omega'_x \omega'_z = M'_{Oy} \\ J'_z \dot{\omega}'_z + (J'_y - J'_x)\omega'_x \omega'_y = M'_{Oz} \end{matrix} \tag{7}$$

The dynamics monographs show that the system (7) can be analytically integrated in three situations: 1) the Euler Poinsot case $\boldsymbol{M}'_a = 0$; 2) the Lagrange Poisson case $J'_x = J'_y$; 3) the Sofia Kovalevskaia case, the center of mass of the rigid should be in the plane $Ox'y'$; $J'_x = J'_y = 2J_3$.

An axi-symmetric rigid body but without rotational symmetry is considered, having a fixed point placed on the axis of symmetry, as in figure 4 actuated by its own weight $\boldsymbol{G}$; the center of gravity is placed on the axis of symmetry at a distance ζ from the fixed point. The single exterior torque acting upon the gyroscope is the one produced by the gravity force $\boldsymbol{G}$, expressed by the projections on the axes of the mobile frame:

$$\boldsymbol{M}_O = Mg\zeta \sin\theta \left[\cos\varphi \quad \sin\varphi \quad 0\right]^T \tag{8}$$

3. Numerical integration of the equations of motion

The values of the angles ψ, θ, φ and their derivatives must be stipulated for the initial instant, in order to integrate numerically the system (7).

For the derivatives calculus, the relation (6) of definition of angular velocity is recalled. The versors $\boldsymbol{k}$ and $\boldsymbol{n}$ are expressed in the mobile frame:

$$\begin{bmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{bmatrix} = \begin{bmatrix} \sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ \cos\theta \end{bmatrix}\dot{\psi} + \begin{bmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{bmatrix}\dot{\theta} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\dot{\varphi} \tag{9}$$

Using the expressions of the angular velocity components expressed as functions of Euler's angles, their derivatives with respect to time and applying the relations (7), a system of three differential equations of second degree is obtained. In order to solve this type of system, the introduction of an auxiliary variable is required for obtaining a system of six linear differential equations of first order. The equations (9) permit writing the derivatives of Euler's angles as functions of the components of the angular velocity on the axes of the mobile system.

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} \sin\varphi / \sin\theta & \cos\varphi / \sin\theta & 0 \\ \cos\varphi & -\sin\varphi & 0 \\ -\sin\varphi\cos\theta / \sin\theta & -\cos\varphi\cos\theta / \sin\theta & 1 \end{bmatrix}\begin{bmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{bmatrix} \tag{10}$$

From relations (8) and (9) it is obtained:

$$\begin{bmatrix} \dot{\omega}'_x \\ \dot{\omega}'_y \\ \dot{\omega}'_z \end{bmatrix} = \begin{bmatrix} \omega'_y \omega'_z \left(J_y - J_z\right)/J_x \\ \omega'_z \omega'_x \left(J_z - J_x\right)/J_y \\ \omega'_y \omega'_y \left(J_x - J_y\right)/J_z \end{bmatrix} + Mg\zeta \sin\theta \begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{bmatrix} \tag{11}$$

The equations (10) and (11) can now be regarded as a system of six differential nonlinear equations of unknowns $\psi, \theta, \varphi, \omega'_x, \omega'_y$ and ω'_z. A numerical method must be applied for the integration of this system [4] and the stipulation of the values of the six unknowns for the initial moment is required.

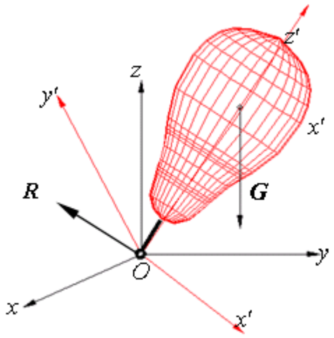

Figure 3. The rigid body, with a fixed point, with rotational symmetry

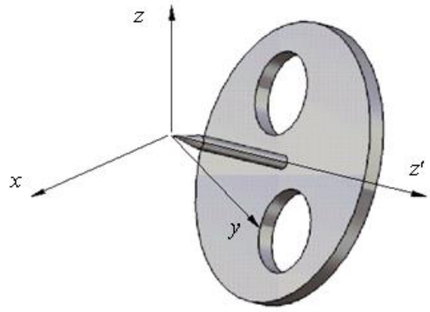

Figure 4. Gyroscope with axial symmetry without rotational symmetry

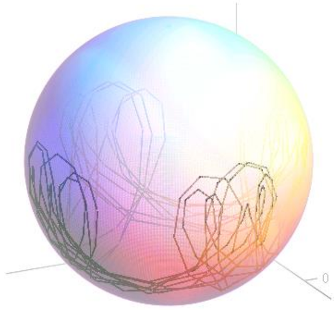

Figure 5. Spatial trajectory of the centre of mass of the gyroscope

$$
\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\varphi} \\ \dot{\omega}'_x \\ \dot{\omega}'_y \\ \dot{\omega}'_z \end{bmatrix} = \begin{bmatrix} \omega'_x \sin\varphi / \sin\theta + \omega'_y \cos\varphi / \sin\theta \\ \omega'_x \cos\varphi - \omega'_y \sin\varphi \\ -\omega'_x \sin\varphi \cos\theta / \sin\theta - \omega'_y \cos\varphi \cos\theta / \sin\theta + \omega'_z \\ \omega'_y \omega'_z \left(J_y - J_z\right) / J_x + Mg\zeta \sin\theta \cos\varphi \\ \omega'_z \omega'_x \left(J_z - J_x\right) / J_y + Mg\zeta \sin\theta \sin\varphi \\ \omega'_x \omega'_y \left(J_x - J_y\right) / J_z \end{bmatrix} \tag{12}
$$

The system (12) is the general system of equations which describes the motion of any type of gyroscope. Next, a gyroscope that has the Oz' axis as principal axis of inertia is considered, with the particularity that the moments of inertia with respect to the other principal axes are not equal anymore, as in the cases of integrability. Such a gyroscope can be obtained practically from a gyroscope with axial rotational symmetry, by removing some material symmetrically with respect to the axis of symmetry, as shown in figure 5. As it can be noticed from the relation (12), a first consequence of the asymmetry $J_x \neq J_y$ is the fact that the rotation motion of the gyroscope about the axis of symmetry is no longer a rotation with constant velocity. In figure 5 is presented the position of the centre of mass of the gyroscope.

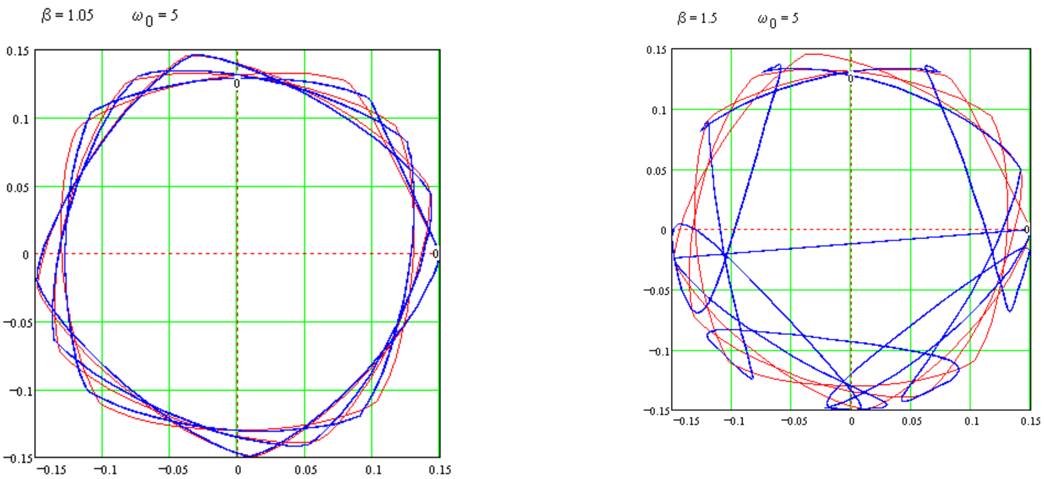

Figure 6. The effect of the geometrical asymmetry β upon the trajectory of the centre of mass

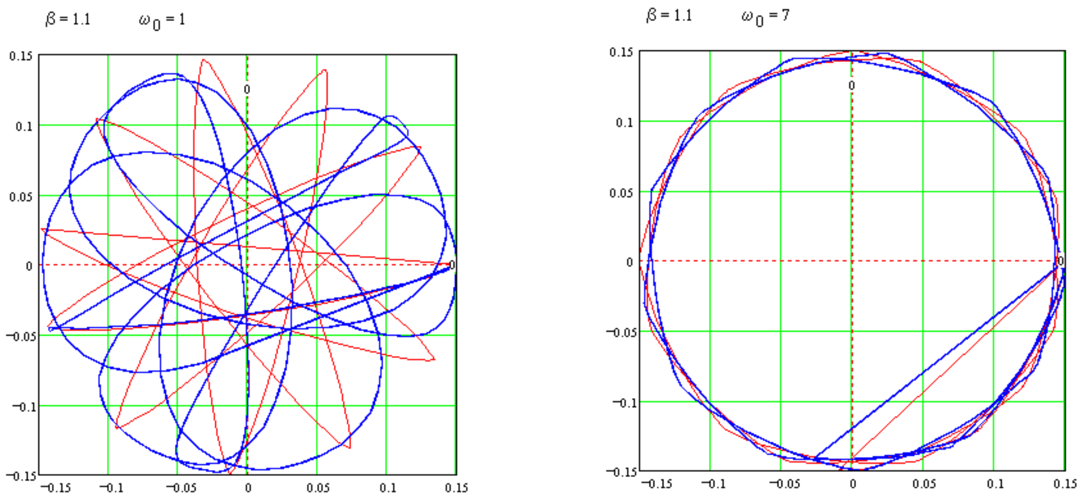

Figure 7. The effect of the own initial angular velocity upon the trajectory of the centre of mass

In order to emphasize the effect of lack of rotational symmetry, the traces of the trajectory of the center of mass on the fixed plane Oxy are presented for different values of the asymmetry coefficient $\beta = J_y / J_x$ in figure 6 and as function of the own initial angular velocity, in figure 7, respectively. The plots from figure 6 and figure 7 present comparatively the graphs for the gyroscope with rotational symmetry (red traces) and for the gyroscope without rotational symmetry (blue traces).

4. Conclusions

The paper presents the method of obtaining the finite equations for a symmetrical gyroscope but without rotational symmetry. First, the system of differential equations of the gyroscope under the form of a nonlinear system of six differential equations of first order is obtained; the unknowns of the system are the derivatives of the Euler's angles and the derivatives of the projections of angular velocity on the axes of the coordinate frame attached to the gyroscope.

The system obtained is valid for the motion of any type of gyroscope. In order to illustrate the effect of asymmetry upon the motion of the gyroscope, the projections of the trajectories on the horizontal plane for the gyroscope with rotational symmetry and for the one without rotational symmetry were presented comparatively. The graphical representations were made firstly for different values of the coefficient of asymmetry but for the same initial conditions and secondly, for the same asymmetry but for different values of the own angular velocity.

As anticipating, the differences diminish with increasing angular velocity, fact easy to perceive qualitatively. The paper permits achieving quantitative aspects concerning the differences between the motions from the two cases.

5. References

[1] Brandt L 1947 *Vector and Tensor Analysis* (John Willey and Son)

[2] Voinea R, Voiculescu D and Simion F P 1989 *Introduction to Solid Mechanics (Introducere în mecanica solidului* in Romanian) Ed. Academiei Romane Bucuresti)

[3] McCarthy J M 1990 *Introduction to Theoretical Kinematics* (MIT Press)

[4] Hamming R W 1987 *Numerical Methods for Scientists and Engineers* 2nd Revised ed. (Dover Books on Mathematics)

Acknowledgement

This work was supported by a grant of the Romanian Ministry of Research and Innovation, CCCDI – UEFISCDI, project number PN-III-P1-1.2-PCCDI-2017-0404 / 31PCCDI/2018, within PNCDI III.